Ruled surfaces

Def ruled surface: $\pi : \underset{\text{surface}}{X} \twoheadrightarrow \underset{\text{ns curve}}{B}$

s.t. 1) $\forall p \in B \quad X_p \cong \mathbb{P}^1$

2) $\exists \sigma : B \to X \quad \pi \circ \sigma = id_B$ section

Example $B \times \mathbb{P}^1 \xrightarrow[pr_1]{} B$

Lemma Any two fibres of a ruled surface are numerically equivalent.

Proof $F = X_p$, $F' = X_{p'}$ two fibres

By Bertini, suffices to show $(F.C) = (F'.C)$ for all $C \overset{\subseteq}{i} X$ nonsingular irred.

The morphism $\pi \circ i : C \to B$ is finite or constant

if $\pi \circ i$ constant, then $C = X_{\pi(C)}$ fibre.

if $C \neq F, F'$, then $(C.F) = 0 = (C.F')$

if $C = F$, then $(C.F - F') = (F.F) - (F.F') = (F.F)$

$$= \deg_F (\underbrace{\mathcal{O}_X(F)|_F}_{(\pi \circ i)^*(\mathcal{O}_B(p)) \cong \mathcal{O}_F}) = 0$$

if $\pi \circ i$ finite, then

$$\deg_C(\mathcal{O}_X(F - F')|_C) = \deg_C((\pi \circ i)^* \mathcal{O}_B(p - p'))$$

$$= \deg(\pi \circ i) \cdot \underbrace{\deg(\mathcal{O}_B(p - p'))}_{=0} = 0 \qquad \square$$

<u>Lemma</u> 1) F_0 fibre of π, $D \in \mathrm{Div}(X)$ s.t. $(D.F) \geq 0$

Then $\pi_* \mathcal{O}(D)$ is locally free of rank $(D.F)+1$

2) $\pi_* \mathcal{O}_X \cong \mathcal{O}_B$

<u>Proof</u> 1) all fibres $F = X_p$ are $\mathbb{P}^1$'s

$$\Rightarrow H^0(\mathcal{O}_X(D)|_{X_p}) = \deg(\mathcal{O}_X(D)|_{X_p}) + 1$$

$$= (D.X_p) + 1 = (D.F)+1 \quad \forall p \in B$$

$$H^1(\mathcal{O}_X(D)|_{X_p}) = 0 \qquad \forall p \in B$$

Cohomology and base change

$\Rightarrow \pi_* \mathcal{O}_X(D)$ locally free $(D.F)+1$

2) $\mathcal{O}_B \longrightarrow \pi_* \mathcal{O}_X$

$$\forall p \in B : \quad \pi_* \mathcal{O}_X \otimes k(p) \xrightarrow{\cong} H^0(X_p, \mathcal{O}_{X_p}) \underset{k}{\overset{}{\rotatebox{90}{$\cong$}}}$$

$$\Rightarrow \mathcal{O}_{B,p} \longrightarrow \pi_* \mathcal{O}_{X,p}$$

$$1 \longmapsto \text{generates} \qquad \square$$

<u>Projectivisations of vector bundles.</u>

S scheme, $\mathcal{E}$ locally free sheaf on S

$$\mathbb{P}(\mathcal{E}) = \underline{\mathrm{Proj}}_S(\mathrm{Sym}(\mathcal{E}^\vee)) \longrightarrow S$$

Similar to projective space, this satisfies a universal property

$$\mathrm{Hom}_{Sch/S}\left(\begin{smallmatrix} X \\ \pi\downarrow \\ S \end{smallmatrix}, \begin{smallmatrix} \mathbb{P}(\mathcal{E}) \\ \downarrow \\ S \end{smallmatrix}\right) = \left\{ (\mathcal{L}, \pi^*\mathcal{E} \twoheadrightarrow \mathcal{L}) \;\middle|\; \mathcal{L} \in \mathrm{Pic}(X/S) \right\}$$

Example $\mathcal{E}$ locally free sheaf of rank 2 on B

Then $\mathbb{P}(\mathcal{E}) \to B$ is a ruled surface.

Proof fibres $= \mathbb{P}^1$: $\mathbb{P}(\mathcal{E})_b \cong \mathbb{P}(k^{\oplus 2}) = \mathbb{P}^1$

existence of section:

let $U \subset B$ s.t. $\mathcal{E}|_U \cong \mathcal{O}_B|_U^{\oplus 2}$

Then $\mathbb{P}(\mathcal{E})|_U \cong \mathbb{P}(\mathcal{O}_U^{\oplus 2}) = U \times \mathbb{P}^1$

has a section $u \mapsto (u, \infty)$

$B \setminus U$ = finite set of points

use valuative criterion of properness of $\mathbb{P}(\mathcal{E})$

to extend to a section $B \to \mathbb{P}(\mathcal{E})$

$\square$

Theorem B ns curve.

$$\left\{ \begin{array}{c} \text{locally free sheaves } \mathcal{E} \\ \text{of rank 2 over } B \end{array} \right\} / \mathrm{Pic}(B) \to \left\{ \begin{array}{c} \text{fibered surfaces} \\ X \to B \end{array} \right\} /_{\cong_B}$$

$$\mathcal{E} \longmapsto \mathbb{P}(\mathcal{E})$$

is a bijection.

Proof $\pi : X \to B$ ruled surface

$\sigma : B \to X$ section , $C_0 := \sigma(B)$

$(C_0 . F) = \deg_{C_0}(\mathcal{O}_X(F)|_{C_0})$ $\qquad F = X_p$

$= \deg_B(\sigma^* \pi^* \mathcal{O}_B(p))$

$= \deg_B(\mathcal{O}_B(p)) = 1$

Lemma

$\Rightarrow$ $\mathcal{E} := \pi_* \mathcal{O}_X(C_0)$ rank 2 loc. free sheaf

adjunction map $\alpha: \pi^* \mathcal{E} = \pi^* \pi_* \mathcal{O}_X(C_0) \to \mathcal{O}_X(C_0)$

α is surjective

indeed, by Nakayama's lemma it suffices to check on fibres (i.e., whether the cokernel is nonzero or not)

$X_p \cong \mathbb{P}^1$, $\deg(\mathcal{O}_X(C_0)|_{X_p}) = (C_0 . X_p) = 1$

$\Rightarrow$ $\mathcal{O}_X(C_0)|_{X_p}$ generated by global sections

Cohomology and base change

$\Rightarrow$ $\mathcal{E} \otimes k(p) \twoheadrightarrow H^0(\mathcal{O}_X(C_0)|_{X_p})$ surjective

$\Rightarrow$ α surjective

By universal property of projectivisation, get morphism

$\varphi: X \longrightarrow \mathbb{P}(\mathcal{E})$ over B

s.t. $\varphi^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_X(C_0)$

$\mathcal{O}_X(C_0)|_{X_p}$ very ample.

$\Rightarrow$ $X_p \hookrightarrow \mathbb{P}(\mathcal{E})_p$ closed embedding

$\Rightarrow$ $\varphi: X_p \xrightarrow{\cong} \mathbb{P}(\mathcal{E})_p$ isomorphism on fibres

Zariski's Main theorem $\Rightarrow$ φ isomorphism. $\square$

Cor: Ruled surfaces over B are birational to $B \times \mathbb{P}^1$

Pf: $\mathbb{P}(\mathcal{E}) \to B$ ruled surface

$U \subset B$ open s.t. $\mathcal{E}|_U \cong \mathcal{O}_B^{\oplus 2}$

$\Rightarrow \mathbb{P}(\mathcal{E})|_U \cong \mathbb{P}(\mathcal{O}_B^{\oplus 2})|_U \cong \mathbb{P}^1 \times U$ $\square$

Prop $\pi : X \to B$ ruled surface ; $C_0 = \sigma(B)$

$\Rightarrow$ $\mathrm{Pic}(X) \cong \mathbb{Z} C_0 \oplus \mathrm{Pic}(B)$

$\mathrm{Num}(X) \cong \mathbb{Z} C_0 \oplus \mathbb{Z} F$

w/ $(C_0 . F) = 1$, $(F . F) = 0$

Proof:

$D \in \mathrm{Div}(X)$

$D' := D - (D.F) C_0$

$(D'.F) = (D.F) - (D.F)(C_0.F) = 0$

Lemma $\Rightarrow$ $\pi_* \mathcal{O}_X(D')$ invertible sheaf on B

and $\mathcal{O}_X(D') \cong \pi^* \pi_* \mathcal{O}_X(D')$ (Coh BC)

$\Rightarrow$ $\mathrm{Pic}(X) = \langle C_0, \pi^* \mathrm{Pic}(B) \rangle$

$\mathrm{Pic}(B) \xrightarrow{\pi^*} \mathrm{Pic}(X)$ has cosection.

$\mathrm{Pic}(X) \xrightarrow{\sigma^*} \mathrm{Pic}(B)$ $\sigma^* \circ \pi^* = id.$

$\Rightarrow$ $\mathrm{Pic}(X) \cong \underbrace{\mathrm{Pic}(X)/\mathrm{Pic}(B)}_{= \langle C_0 \rangle} \oplus \mathrm{Pic}(B)$

$n(C_0.F) = (nC_0.F) \Rightarrow nC_0 \neq mC_0$ for $n \neq m$. $\square$

<u>Corollary</u>: $p_a(X) = g(B)$, $p_g(X) = h^2(X, \mathcal{O}_X) = 0$

$q(X) := h^1(X, \mathcal{O}_X) = g(B)$

<u>Pf</u> $\pi_* \mathcal{O}_X = \mathcal{O}_B$

$\Rightarrow h^i(\mathcal{O}_X) = h^i(\mathcal{O}_B)$, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_B)$

$\Rightarrow p_a(X) = 1 + \chi(\mathcal{O}_X) = g(B)$ $\square$.

Prop $\pi : X \to B$ ruled surface

1) $\exists\, \mathcal{E}$ rank 2. loc. free on B s.t.

- $X \cong_B \mathbb{P}(\mathcal{E})$
- $H^0(\mathcal{E}) \neq 0$
- $\forall\, \mathcal{L} \in \mathrm{Pic}(C)$, $\deg(\mathcal{L}) < 0$ $\quad H^0(\mathcal{E} \otimes \mathcal{L}) = 0$

2) $-e = \deg(\mathcal{E}) = \deg(\Lambda^2 \mathcal{E})$ is invariant of X/B

3) $\exists\ \sigma : B \to X$ s.t. $\mathcal{O}_X(B_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$

Proof $X \cong \mathbb{P}(\mathcal{E}')$ for some $\mathcal{E}'$ rank 2 loc. free.

for $\deg(\mathcal{M}) \gg 0$ have $H^0(\mathcal{E}' \otimes \mathcal{M}) \neq 0$

Main idea: $\sigma : B \longrightarrow \mathbb{P}(\mathcal{E}')$ corresponds to $\mathcal{E}' \twoheadrightarrow \mathcal{L}$ on B so have ses.

$$0 \to \mathcal{N} \to \mathcal{E}' \to \mathcal{L} \to 0 \quad \text{on } B$$

$\Rightarrow$ for $\deg(\mathcal{M}) \ll 0$ $\quad H^0(\mathcal{E}' \otimes \mathcal{M}) = 0$

Choose $\mathcal{M}$ s.t.

$$\deg(\mathcal{M}) = \min \left\{ \deg(\mathcal{M}') \;\middle|\; \begin{array}{l} \mathcal{M}' \text{ s.t.} \\ H^0(\mathcal{E}' \otimes \mathcal{M}') \neq 0 \end{array} \right\}$$

$\Rightarrow \mathcal{E} = \mathcal{E}' \otimes \mathcal{M}$ does the job.

$\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{M})$

$\Rightarrow$ invariance follows.

for 3): pick $s \in H^0(\mathcal{E}) \setminus 0$

have SES $\quad 0 \to \mathcal{O}_B \xrightarrow{s} \mathcal{E} \xrightarrow{q} \mathcal{L} \to 0$

show that $\mathcal{L}$ is a line bundle,

$$\mathcal{O}_C \not\cong \mathcal{F} = \ker\left(\mathcal{E} \to \mathcal{L} \to \mathcal{L}^{\text{tor}} \to 0\right) \subseteq \mathcal{E}$$

if ${}^{\text{tor}}\neq 0$ tor. free rank 1 on B $\Rightarrow$ $\mathcal{F}$ invertible.

$$H^0(\mathcal{E} \otimes \mathcal{F}^\vee) \neq 0$$

$$\deg(\mathcal{F}^\vee) = -\deg(\mathcal{F}) < 0 \qquad \text{b/c } \mathcal{L} \not\cong \mathcal{O}_C$$

$\Rightarrow$ $\mathcal{E} \twoheadrightarrow \mathcal{L}$ determines desired section $\square$.

Terminology $\mathcal{E}$ as in Proposition is called normalized

Proposition $\pi : \mathbb{P}(\mathcal{E}) \to B$

$\mathcal{E}$ normalized, $\mathcal{O}(B_0) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$

$$\underset{\text{section}}{\sigma : B \to \mathbb{P}(\mathcal{E})} \iff \mathcal{E} \twoheadrightarrow \mathcal{L}$$

$$\Rightarrow \deg_B(\mathcal{L}) = (\sigma(B) . B_0)$$

and $\mathcal{O}_X(\sigma(B)) \cong \mathcal{O}_X(B_0) \otimes \pi^*(\mathcal{L} - \det(\mathcal{E}))$

In particular $B_0^2 = -e$

Proof $\mathcal{L} = \sigma_0^* \pi^* \mathcal{L} = \sigma_0^* \mathcal{O}_X(D)$

$$\Rightarrow \deg(\mathcal{L}) = \deg_{B_0}(\mathcal{O}_X(D))|_{B_0} = (B_0 . D)$$

SES: $0 \to \mathcal{N} \to \mathcal{E} \to \mathcal{L} \to 0$

$$\Rightarrow \mathcal{O}(B_0 - D) \cong \pi^* \mathcal{N} \qquad \square.$$